Correlation issues in structural models : the example of default swaps on two credit instruments.

Capital structure arbitrage
Evry, 16-17 Juin 2005

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Correlation products: modeling, theoretical and computational issues

The valuation of correlation products such as STCDOs (giving protection on given "tranches" such as 3%-6% ["attachment points"] of cumulated losses on large porfolios of default-prone credit instruments: corporate bonds, CDS...) and similar credit derivatives shares with equity-to-credit models the property of relying largely on firm's value models.

Most classical illustration (market standard for the mark-to-market calibration of correlation from observed prices of STCDOs): one-factor gaussian copula (OGC) model. A Merton-type model: $N >> 0$ obligors; the default of an obligor occurs when the value of its assets fall below a barrier (calibrated on CDS spreads). Problem: how to take into account the correlation of defaults ?

(After a suitable normalization) it is assumed that the values of the assets of the obligors are driven by a common standard normally distributed variable Z and normally distributed idiosyncratic factors Z_i:

$$V_n(T) = \varrho Z + \sqrt{1-\varrho^2} Z_i.$$

Conditionnally on Z the V_n are independent, making the model numerically tractable. Various variants: multifactor models...

However, several shortcomings of these classical models:

- Correlation is not constant.

  Fact: implied correlation computed from observed STCDO prices depend on the attachment points of the tranches (correlation skews phenomenon).

Fundamental explanation to this behavior of correlation:

"Because CDOs are sensitive to correlation, and correlation of defaults is typically driven by the business cycle, the correlation risk of CDO tranches can be characterized, and measured, as "business cycle risk"... For example, mezzanine tranches are leveraged bets on business cycle risk" (Bank Int. Settl.)

Main issue in correlation modeling: quantify market views on the business cycle risk; incorporate to the pricing methodology; understand and calibrate correlation skews accordingly. Leads to a stochastic approach to correlation (Zeliade's model for CDOs is based on these ideas.)

- Dynamic modeling. The Merton model of the OGC approach is a static model. It can be accomodated partially to the dynamics of CDS spreads on the various obligors (semi-dynamic copula model); does not give a consistent specification of underlying CDS spreads dynamics.

Main issue: computational, need for new ideas/new tools, even for a small number of obligors.

- Recovery rates on the various obligors should be stochastic (more on this point later).

Issues on which we will concentrate: impact of correlation on recovery rates and computational problems from the dynamic modeling point of view. Little was known on the last topic (He-Keirstead-Rebholz, Zhou); the analytic approach gives little insight on the problem and leads very fast to untractable computations. Geometrical/probabilistic approach should change the whole picture.

Model of Lardy-Finkelstein : CreditGrades.

"Robust but simple framework linking the credit and equity markets". Structural, Black-Cox, model : equity = call option on the assets V_t of a firm. The firm defaults when the assets fall below a barrier depending on the value of the debt and on the recovery rate.

(Very nice) specificity of the model: explains the short term spreads on bonds (classical limit of structural models).

Various classical ways for modeling short term spreads : jump processes, calibration of default barrier on market spreads. LF have introduced a random behavior of the barrier, that they interpret as a random behavior of recovery rates.

Model:

$$\frac{dV_t}{V_t} = \sigma dW_t + \mu dt$$

Hypothesis: steady level of leverage (the debt has the same drift as the stock price): $\mu = 0$.

Debt: D, Recovery rates lognormal with mean $\overline{L}$ and standard deviation λ:

$$L = \overline{L} e^{\lambda Z - \frac{\lambda^2}{2}}$$

Default occurs when assets fall below LD:

$$V_0 e^{\sigma W_t - \frac{\sigma^2 t}{2}} > \overline{L} D e^{\lambda Z - \frac{\lambda^2}{2}}.$$

Simplifying assumption: LF consider the process

$$X_t := \sigma W_t - \lambda Z - \frac{\sigma^2 t}{2} - \frac{\lambda^2}{2}$$

as the $t \geq 0$ values of a BM starting in $-\Delta t = \frac{-\lambda^2}{\sigma^2}$, drift $\frac{-\sigma^2}{2}$, variance σ^2.

The barrier condition now reads:

$$X_t > log(\frac{\overline{L}D}{V_0}) - \lambda^2, \ t \in [-\Delta t, T].$$

Survival probability computed as in Black-Cox:

$$\mathbf{P}'(t) = \Phi(\frac{-A_t}{2} + \frac{log(d)}{A_t}) - d\Phi(-\frac{A_t}{2} - \frac{log(d)}{A_t}).$$

$$d = \frac{V_0 e^{\lambda^2}}{\overline{L}D}, \quad A_t = \sqrt{\sigma^2 t + \lambda^2}.$$

Correlation in a LF model on several assets

Correlation on asset values $V_t^i, i = 1, 2$ of OGC type:

$$\frac{dV_t^i}{V_t^i} = \sigma^i dW_t^i,$$

$$Cov(W_t^1, W_t^2) = \varrho t.$$

How about recovery rates ?

Statistic studies and basic economic intuition show that:

-Recovery rates are correlated

-They are negatively correlated to the frequency of defaults

-The level of recovery rates is driven by the same fundamentals than defaults, namely economic cycles (which are approximated by the common gaussian factor in OGC models).

Reasonable assumptions (to get a robust, simple and tractable model, as in the single-name case):

$\Delta t^1 = \Delta t^2$: same (relative) level of uncertainty on recovery rates.

We take asset correlation as a proxy for the correlation of recovery rates (equivalently, we assume that recovery rates have the same statistical dependency as assets on macroeconomical [resp. sectorial if the firms belong to the same sector] fundamentals).

Pricing methodology

Problem : how to price a correlation product on two credit instruments as a 1rst/2nd-to-default swap ?

Single name case: closed formulas for credit derivatives follow from the reflection principle and the strong Markov property of BM (see e.g. Bielecki-Rutkowski, Credit Risk: Modeling, Valuation and Hedging.).

Dimension $n \geq 2$: Problem = compute survival probabilities, exit times, transition densities for BM(n) in a polyedral cone. Geometry of the cone given by assets correlation matrix.

Known: in dim. 2, partial result: Sommerfeld reflection principle (particular case when angle of the cone $\beta = \frac{\pi}{n}$).

The geometrical approach

Fundamental idea : the planar BM should be replaced by another (new kind of) process, better suited to the problem.

Construction: the planar BM is lifted to the universal covering of $\mathbf{R}^2 - \{0,0\}$. The new paths on $\mathbf{R}^+ \times \mathbf{R}$ are projected onto the (locally euclidean) quotient space $\mathbf{R}^+ \times \frac{\mathbf{R}}{2\beta}$.

It follows from the classical theory of paths and covering spaces that paths on $\mathbf{R}^2 - \{0,0\}$ are (essentially) in bijection with paths on $\mathbf{R}^+ \times \frac{\mathbf{R}}{2\beta}$. Moreover all the properties of the Brownian motion that rely only on local arguments (where local means with respect to time and space simultaneously) also hold for the new process written $\mathbf{X}_\beta(t)$ on the state space $\mathbf{R}^+ \times \frac{\mathbf{R}}{2\beta}$.

The classical proofs hold *mutatis mutandis*. For example, the density of $\mathbf{X}_\beta(t+\epsilon)$, $0 < \epsilon << 1$ at $\mathbf{z} + re^{i\zeta}$ for $r << ||\mathbf{z}||$ conditional to $\mathbf{X}_\beta(t) = \mathbf{z}$ is given by:

$$f(\mathbf{z}, \mathbf{z} + re^{i\zeta}, \epsilon) = \frac{1}{2\pi\epsilon} e^{-\frac{r^2}{2\epsilon}}.$$

Similarly, the transition probability densities

$$f((\rho, \theta), (\mu, \kappa), t)$$

satisfy the Kolmogorov forward equation:

$$\frac{\partial f}{\partial t} = \frac{1}{2}[\frac{\partial^2 f}{\partial \mu^2} + \frac{1}{\mu}\frac{\partial f}{\partial \mu} + \frac{1}{\mu^2}\frac{\partial^2 f}{\partial \kappa^2}],$$

where (ρ, θ) and (μ, κ) belong to $\mathbf{R}^+ \times \mathbf{R}/2\beta$, with the initial condition:

$$f((\rho, \theta), (\mu, \kappa), 0) = \delta_{(\rho,\theta)=(\mu,\kappa)}.$$

Theorem 1 *The transition probability densities* $f((\rho, \theta), (\mu, \kappa), t)$ *for the process* $\mathbf{X}_\beta$ *are* *given by:*

$$\frac{1}{\beta}\int_0^\infty \lambda e^{-\lambda^2 t}[\frac{1}{2}J_0(\lambda\mu)J_0(\lambda\rho)$$

$$+\sum_{n=1}^{\infty} J_{\frac{n\pi}{\beta}}(\lambda\mu)J_{\frac{n\pi}{\beta}}(\lambda\rho)cos(\frac{n\pi}{\beta}(\kappa-\theta))]d\lambda.$$

This formula is a natural generalization to $\mathbf{R}^+\times\mathbf{R}/2\beta$ of the classical formula for heat conduction, when expanded in terms of Bessel functions. Indeed, let $\beta=\pi$. Then, $\mathbf{X}_\beta$ identifies with the standard Brownian motion. Considering the heat equation $\frac{\partial f}{\partial t}=\frac{1}{2}\Delta f$, the temperature at (μ,κ) at t due to an instantaneous unit source at (ρ,θ) at $t=0$ is given by:

$$\frac{1}{2\pi t}e^{-\frac{R^2}{2t}}=\frac{1}{2\pi}\int_0^\infty \lambda e^{-\frac{\lambda^2 t}{2}}J_0(\lambda R)d\lambda$$

(Weber's first integral), where $R^2=\rho^2+\mu^2-2\rho\mu cos(\kappa-\theta)$. The identification with the formula in the Proposition when $\beta=\pi$ follows from Neumann's expansion:

$$J_0(\lambda R) = J_0(\lambda\rho)J_0(\lambda\mu)$$

$$+2\sum_{n=1}^{\infty} J_n(\lambda\rho)J_n(\lambda\mu)cosn(\theta-\kappa).$$

Conclusion for credit derivatives: in a dynamical setting, no hope to get closed pricing formula for correlation products avoiding special functions. On the other hand, formulas obtained can look weird at the first sight but, in the end, should be well adaped to a numerical treatment.

The generalized reflection principle.

Theorem 2 *(Generalized reflection principle) We have:*

$$\mathbf{P}(\mathbf{X}^{\mathbf{x}}(T) \in d\mu e^{id\kappa}, \tau \leq T) = \mathbf{P}(\mathbf{X}_{\beta}(T) \in d\mu d\kappa, \tau \leq T)$$

$$= \mathbf{P}(\mathbf{X}'_\beta(T) \in d\mu \cdot d\kappa),$$

Here, $\mathbf{X}^{\mathbf{x}}$ is a planar BM starting from $\mathbf{x}$, τ is the exit time from a cone (angle β), $\mathbf{X}_\beta$ is the lift of $\mathbf{X}^{\mathbf{x}}$ to $\mathbf{R}^+ \times \frac{\mathbf{R}}{2\beta}$ and $\mathbf{X}'_\beta$ is the reflexion of $\mathbf{X}_\beta$ along the barrier in the new state space.

Corollary: all the computations, all the single name pricing formulas that rely on the reflection principle and the stong Markov properties of BM(1) translate automatically into formulas for derivatives on two correlated assets, replacing BM(1) by the new processes.

Pricing of a digital swap on two credit instruments.

We assume that the defaults of the underlyings are driven by the 2-dim generalization of the Lardy-Finkelstein model.

We assume eg that the swap S is a first-to-default of digital type: the payoff (say 1) is settled at the maturity date T; the protection buyer pays (continuously) a spread s. We have to solve for s so that the expected premium payments equal the expected loss payouts.

Parameters for the underlyings are as in the beginning, r is the (constant) riskfree interest rate.

Theorem 3 *The price (the spread s) of the digital-type swap S is given by:*

$$s = r \frac{A}{B + C - A},$$

where, for example:

$$A = e^{-rT} \int_0^\infty \int_0^\beta \frac{2\mu}{\beta(T + \Delta t)} e^{<\vec{\phi}|\vec{\mu}-\vec{\rho}> - \frac{||\vec{\phi}||^2 (T+\Delta t)}{2}}$$

$$\sum_{n=1}^{\infty} e^{-\frac{\rho^2+\mu^2}{2(T+\Delta t)}} sin(\frac{n\pi\theta}{\beta}) sin(\frac{n\pi\kappa}{\beta}) I_{\frac{n\pi}{\beta}}(\frac{\rho\mu}{T+\Delta t}) d\mu d\kappa.$$

and where:

$$y^i := log(\frac{\overline{L}^i D^i}{V_0^i}) - (\lambda^i)^2, \; i = 1, 2;$$

$$\nu_i := \frac{\sigma_i^2}{2},$$

$$\beta = \frac{\pi}{2} + arcsin(\varrho)$$

$$\phi_1 := \frac{\nu_1\sigma_2 - \nu_2\sigma_1\varrho}{\sigma_1\sigma_2\sqrt{1-\varrho^2}}$$

$$\phi_2 := \frac{\nu_2}{\sigma_2}$$

$$\rho e^{i\theta} = \frac{y^1\sigma_2 - sin(2\alpha)y^2\sigma_1}{\sigma_1\sigma_2\sqrt{1-\varrho^2}} + i\frac{y^2}{\sigma_2}$$

B and C are also given by double integrals of Bessel function (structurally slightly different, but of the same numerical complexity as A).

Proof : generalized reflection principle + local properties of BM(2) + Girsanov.